ABSTRACT

A classical approach to interpolation of sampled data is polynomial interpolation. However, from the sampling theorem it follows that the ideal approach to interpolation is to convolve the given samples with the sinc function. In this paper we study the properties of the sinc-approximating kernels that can be derived from the Lagrange central interpolation scheme. Both the finite-extent properties and the convergence property are analyzed. The Lagrange central interpolation kernels of up to ninth order are compared to cardinal splines of corresponding orders, both by spectral analysis and by rotation experiments on real-life test-images. It is concluded that cardinal spline interpolation is by far superior.

1. INTRODUCTION

In many digital image processing applications, such as registration, segmentation, or visualization, images frequently need to be enlarged, reduced, or geometrically transformed. All of these operations require interpolation. In the classical literature, the customary approach to interpolation of “sampled data” has always been to fit a single polynomial through the given data, in which case the order of the polynomial interpolant is directly determined by the number of samples incorporated. A well-known example of this is Lagrange interpolation. From the Whittaker-Shannon sampling theorem, on the other hand, it follows that the ideal approach to interpolation is to compute the cardinal series expansion of the given data, which implies convolving the samples with the sinc function.

In this paper, the properties of classical polynomial interpolation, notably Lagrange interpolation, are studied from a digital image processing point of view. After having presented some historical and theoretical background information, we focus on the Lagrange central interpolation scheme, which can be viewed as a digital filtering operation. We study the properties of the sinc-approximating kernels that can be derived from this interpolation scheme. Both the finite-extent properties and the convergence property are analyzed. Finally, the Lagrange central interpolation kernels of up to ninth order are compared to cardinal splines of corresponding orders, both by spectral analysis and by rotation experiments on real-life test-images.

2. CLASSICAL POLYNOMIAL INTERPOLATION

Using classical polynomial interpolation formulae, an interpolant is expressed either in terms of (divided) differences, as proposed by Gregory [1] and Newton [2] in the 17th century, or directly in terms of the sample values, resulting in an interpolation scheme originally due to Waring [3] and Euler [4], but nowadays usually attributed to Lagrange [5]. Since, in principle, all classical polynomial interpolation schemes are equivalent (as follows from the fundamental theorem of algebra), we only consider the one most suitable for the analysis in this paper: Lagrange’s. Furthermore, since these schemes are separable for higher dimensional interpolation problems, we only consider the 1-D case here.

Given $q \geq 2$ equally spaced sample points, $p_k \in \mathbb{Z}(T_s)$, and their corresponding sample values, $I_s(p_k)$, where $T_s$ denotes the inter-sample distance, which we assume to be 1 throughout this paper. In its original form, Lagrange’s interpolation formula results in an analytical function, viz., a polynomial of degree $q-1$. From a signal processing point of view, Lagrange’s interpolation formula may be interpreted in terms of $q-1$ different impulse responses, corresponding to the $q-1$ sample intervals. As has been shown by Schafer & Rabiner [6], all of these impulse responses introduce phase distortions, except in the case of interpolation in the central interval with $q$ being even. Therefore, only the
Lagrange central interpolation formula should be used:

\[ \tilde{I}_L(x) = \sum_{k=k_{\min}}^{k_{\max}} I_s(p_k) L^n_k(x), \quad x \in \mathbb{R}, \]  

(1)

where \( n \) odd is the order to be chosen, \( k_{\min} = -\lfloor n/2 \rfloor \), \( k_{\max} = \lceil n/2 \rceil \), \( p_k = (m + k) \), \( m = \lceil x \rceil \), and the \( L^n_k \) are the so called Lagrange coefficients, defined by

\[ L^n_k(x) \triangleq \prod_{i=k_{\min}}^{k_{\max}, i \neq k} \frac{x - p_i}{p_k - p_i}. \]  

(2)

3. SINC-APPROXIMATING KERNELS

The Lagrange central interpolation formula, Eq. (1), can be rewritten in the form of a convolution [6]:

\[ \tilde{I}_L(x) = \sum_{k=-\infty}^{\infty} I_s(p_k) h_L(x-p_k), \]  

(3)

where \( h_L \) is the \( n \)-th order Lagrange central interpolation kernel, which is obtained by varying \( x \) within a given sample interval while computing \( h_L(x-p_k) = L^n_k(x) \) for all \( k \in [k_{\min}, k_{\max}] \subset \mathbb{Z} \), and by defining it to be zero for all other values of \( k \).

In this section we study the sinc-approximating properties of the Lagrange central interpolation kernel, \( h_L \). Both the finite-extent properties and the convergence property are analyzed.

3.1. Finite-Extent Properties

We note that the Lagrange central interpolation kernel is a piecewise polynomial kernel, which can be written in the form

\[ h_L(x) = \begin{cases} h_j(x), & j \leq |x| < j + 1 \\ 0, & m \leq |x| \end{cases} \]  

(4)

with

\[ h_j(x) \triangleq \begin{cases} h_{j,+}(x), & j \leq x < j + 1 \\ h_{j,-}(x), & -j - 1 < x \leq -j \end{cases} \]  

(5)

and

\[ h_{j,+}(x) \triangleq a_{0j} + a_{1j}x + a_{2j}x^2 + \cdots + a_{nj}x^n \]  

(6a)

\[ h_{j,-}(x) \triangleq a_{0j} - a_{1j}x + a_{2j}x^2 - \cdots - (-1)^n a_{nj}x^n \]  

(6b)

where \( n \) is again the order of the polynomials, \( j = 0, 1, \ldots, m - 1 \), the parameter \( m \in \mathbb{N}\setminus\{0\} \) determines the extent of the kernel, and \( n \) and \( m \) are related by \( n = 2m - 1 \). The \( (n+1)m \) coefficients \( a_{ij} \) may be obtained by evaluating the Lagrange coefficients in the corresponding intervals \( j \leq x < (j + 1), j = 0, 1, \ldots, m - 1 \), that is: \( h_{j,+}(x) = L^n_{j+1}(x-j) \). Equivalently, they can be obtained by imposing the following constraints on the sub-polynomials \( h_{j,+} \):

\[ L1: h_{j,+}(x) = 1 \text{ for } x = 0, \text{ and } j = 0, 1, \ldots, m - 1, \]  

\[ L2: h_{j,+}(x) = 0 \text{ for } x = (j-\lfloor n/2 \rfloor), \ldots, (j+\lceil n/2 \rceil), x \neq 0, \]  

and \( j = 0, 1, \ldots, m - 1, \) which results in a total of \( (n+1)m \) equations that can be solved uniquely.

3.2. Convergence Property

Although the convergence of classical polynomial interpolation schemes to the cardinal series,

\[ \tilde{I}(x) = \frac{\sin(\pi x)}{\pi} \left[ \frac{I_s(p_0)}{x} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{I_s(p_k) + I_s(p_{-k})}{x-p_k} + \frac{I_s(p_0)}{x-p_{-k}} \right], \]  

(7a)

\[ = \sum_{k=-\infty}^{\infty} I_s(p_k) \text{sinc}(x-p_k), \]  

(7b)

has been known by mathematicians since the beginning of this century [7, 8], explicit proofs are not easily found in the image processing literature. Therefore, we have chosen here to present one for Lagrange central interpolation. It is sufficient to prove that

\[ \lim_{n \to \infty} L^n_k(x) = \text{sinc}(x-p_k), \quad \forall x \in \mathbb{R}, \quad \forall k \in \mathbb{Z}. \]  

(8)

By substituting \( p_k = (m + k) \), \( p_i = (m + i) \), \( x = (m + \gamma) \), with \( \gamma = x - \lceil x \rceil \), and \( j = (k-i) \) into Eq. (2), we have

\[ \lim_{n \to \infty} L^n_k(x) = \prod_{j=1 \atop j \neq 0}^{\infty} \left( 1 + \frac{\gamma-j}{j} \right), \]  

(9a)

\[ = \prod_{j=1 \atop j \neq 0}^{\infty} \left( 1 - \frac{(\gamma-k)^2}{j^2} \right). \]  

(9b)

As hinted by Jain [9], Eq. (8) can now be proved by means of Euler’s infinite product expansion of the sine function [10, 11]:

\[ \text{sinc}(x-p_k) \triangleq \frac{\sin(\pi(x-p_k))}{\pi(x-p_k)} = \prod_{j=1}^{\infty} \left( 1 - \frac{(\gamma-k)^2}{j^2} \right). \]  

(10a)

\[ = \prod_{j=1}^{\infty} \left( 1 - \frac{(\gamma-k)^2}{j^2} \right). \]  

(10b)
4. COMPARISON WITH SPLINE INTERPOLATION

A popular alternative approach to polynomial interpolation is spline interpolation, originally introduced by Schoenberg [12]. Although it is never explicitly implemented this way, spline interpolation can be written in a form similar to Eq. (3):

$$\hat{I}_S(x) = \sum_{k=\infty}^{\infty} I_s(p_k) h_S(x - p_k),$$

(11)

where $h_S$ denotes the cardinal spline, defined by

$$h_S(x) = \sum_{k=\infty}^{\infty} (b^n)^{-1} (p_k)^n (x - p_k).$$

(12)

In Eq. (12), the entities $(b^n)^{-1}$ and $\beta^n$ denote the direct and indirect B-spline filters of order $n$, respectively. We note here that it has been shown that cardinal spline interpolation also converges to sinc interpolation [13, 14]. For further details regarding the theoretical and implementational aspects of B-spline interpolation, we refer to Unser et al. [15, 16].

In this section, the performance of the Lagrange central interpolation kernels and the cardinal splines of up to ninth order are compared, both by spectral analysis and by experiments on real-life images.

4.1. Spectral Analysis

The frequency spectrum $\tilde{H}_L$ of the Lagrange central interpolation kernel $h_L$, as well as the spectrum $\tilde{H}_S$ of the cardinal spline $h_S$, for $n = 1, 3, 5, 7, 9$, are shown in Fig. 1. In all plots, the spectrum of the ideal sinc function is shown for comparison.
Figure 2: The four test-images used in the rotation experiment described in Section 4.2. From left to right: Lena, Baboon, Peppers, Vai.

<table>
<thead>
<tr>
<th>Order</th>
<th>Lena</th>
<th>Baboon</th>
<th>Peppers</th>
<th>Vai</th>
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<tr>
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<td>$h_L$</td>
<td>$h_S$</td>
<td>$h_L$</td>
<td>$h_S$</td>
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<td>8.43</td>
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<td>1.72</td>
<td>10.92</td>
<td>7.06</td>
</tr>
</tbody>
</table>

Table 1: Results of the rotation experiment. The figures are the root-mean-square errors (RMSEs) induced by the Lagrange central interpolation kernel, $h_L$, and the cardinal B-spline kernel, $h_S$, for $n = 1, 3, 5, 7, 9$.

From the linear and logarithmic plots in Fig. 1 it is clear that for corresponding orders – except for $n = 1$, in which case the kernels are identical and boil down to the linear interpolation kernel – both the low frequency band-pass and the high frequency band-stop characteristics of the cardinal spline kernel are considerably better than those of the Lagrange central interpolation kernel. This can be explained from the fact that, for any finite order $n > 0$, the latter kernel is not continuously differentiable, i.e., it is an element of the class $C^0$, which entails the presence of considerable high frequency components. The cardinal spline, on the other hand, is a $C^{n-1}$ function, which results in a very smooth interpolant.

4.2. Experiments on Real-Life Images

In order to further analyze the performance of the Lagrange central interpolation kernel as compared to the cardinal spline, we carried out a rotation experiment: each of the four 2-D real-life test-images shown in Fig. 2 was successively rotated over 16 different angles, which added up to a total of 360°. This experiment was repeated for each of the two kernels, for $n = 1, 3, 5, 7, 9$. The accumulated errors made by the different kernels were analyzed by computing the root-mean-square deviation (RMSE) of the grey values in the final processed images relative to the grey values in the corresponding original images. A similar experiment has been used by others [17]. The results of this experiment are summarized in Table 1, which confirm the conclusions drawn from the spectral analysis described in the previous subsection. On average, the RMSEs resulting from spline interpolation were only about 65% of those introduced by Lagrange central interpolation.

5. CONCLUSIONS

In this paper we have studied the properties of the sinc-approximating kernels that can be derived from the Lagrange central interpolation scheme. We have derived kernels of up to ninth order, and it was shown that as the order tends to infinity, the Lagrange central interpolation scheme converges to sinc interpolation. The finite-extent kernels were compared to cardinal splines of corresponding orders, both by spectral analysis and by rotation experiments on 2-D real-life test-images. It was concluded that cardinal spline interpolation is by far superior to Lagrange central interpolation.
6. REFERENCES


